ON OZAWA'S PROPERTY FOR FREE GROUP FACTORS

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Abstract. We give a new proof of a result of Ozawa showing that if a von Neumann subalgebra Q of a free group factor $L\mathbb{F}_n, 2 \leq n \leq \infty$ has relative commutant diffuse (i.e. without atoms), then Q is amenable.

§1. Introduction

The purpose of this note is to present a rather short and elementary proof of the following result of Ozawa:

Theorem 1.1 ([O]). *Let $Q \subset N = L\mathbb{F}_n$ be a von Neumann subalgebra with $Q' \cap N$ diffuse. Then Q is amenable.*

In fact, Ozawa proved the above property for all group factors $N = L\Gamma$ with Γ word-hyperbolic group. The property implies in particular that all such factors N are *prime*, i.e. they cannot be decomposed as a tensor product of II_1 factors. Ge had already proved the primeness of $N = L\mathbb{F}_n$ in ([G]), by using Voiculescu's free entropy techniques ([V]). A combination of results in ([GSh]) and ([J1]) showed that in fact all free products of diffuse von Neumann algebras embeddable into R^ω are prime (see also [J2]). Recently, Peterson used a completely new approach to prove the primeness of group factors $N = L\Gamma$ with Γ satisfying $\beta_1^{(2)}(\Gamma) \neq 0$ ([Pe]). His results cover also the primeness of all free products of diffuse finite von Neumann algebras and of all non-amenable subfactors of $L\mathbb{F}_n$.

Both Ozawa's insight, the use of free entropy yechniques in ([G], [GSh], [J1,2]) and Peterson's new approach are conceptually and technically involved, and aim at showing the property for as many factors as possible.

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In turn, our proof here targets the free group factor case only. The argument is based on the following two properties of $N = L\mathbb{F}_n$:

(1) A "malleability deformation" which allows moving $N = N * \mathbb{C}$ along a path α of automorphisms inside $N * N$, from position $N = N * \mathbb{C}$ to $\mathbb{C} * N$. It is important for us here that the path α be "group-like" and to have a certain "symmetry".

(2) The fact that any $Q \subset N$ which has no amenable summand has "spectral gap" with respect to the orthogonal of N in $N * N$. In other words, there exists a "critical" finite set of unitaries $F \subset Q$ such that if $x \in N * N$ almost commutes with all $u \in F$ then x is almost contained in N.

Property (1) was already considered in ([P1]) and, exactly in the form we need here, on (page 322 of [P2]). A related deformation for free products of arbitrary factors has been used extensively in ([IPeP]). Property (2) is quite evident by one of Connes' criteria for amenability, since $L^2(N * N) \ominus L^2N$ is a multiple of the N bimodule $L^2N\overline{\otimes}L^2N$, of Hilbert-Shmidt operators on L^2N. It may in fact have been noticed before, in some equivalent form. We will nevertheless prove it, for completeness, and in fact also include the proof of (1) from ([P2]).

The proof of Theorem 1.1 then goes as follows: By contradiction we may assume Q has no amenable summand, so by (2) a critical set $F \subset Q$ can be chosen. Thus, any element in $N * N$ that approximately commutes with F in $N * N$ must be almost contained in $N = N * \mathbb{C}$. Due to (pointwise) continuity of α_t, if $t > 0$ is small then $\alpha_t(u)$ is close to u, $\forall u \in F$, so elements that commute with $\alpha_t(F)$ must almost commute with F, thus have to be almost in N. This shows that the unit balls of $\alpha_t(Q' \cap N)$ and $Q' \cap N$ are uniformly close one to another. Known results from ([P1,3] or [PSiSm]) then imply that $Q' \cap N$ and $\alpha_t(Q' \cap N)$ can be "intertwined" with a non-zero partial isometry. The group-like structure of α and its symmetry make possible to "patch" the intertwiners between position t to position $t + \Delta t$, for "incremental" Δt, into a non-zero intertwiner between a diffuse von Neumann subalgebra of $N * \mathbb{C}$ and its image under α_1 in $\mathbb{C} * N$. This in turn contradicts (Corollary 4.3 in [P4]).

Note that this line of proof is identical to the approach in (4.3.2 of [P1]) to Connes-Jones theorem on non-embeddability of property (T) II_1 factors Q_0 into free group factors N. However, that argument showed only that there exist no embeddings $Q_0 \subset N$ with $Q_0' \cap N = \mathbb{C}$, and could not settle the case $Q_0' \cap N$ diffuse, because of a poorer technique for handling the "patching" of the incremental intertwiners in this case: if $Q_0' \cap N$ is diffuse and $\alpha_t(Q_0' \cap N)$ is uniformly close to $\alpha_{t+\Delta t}(Q_0' \cap N)$ then the ensuing intertwiner v_t between these two algebras is a non-zero partial isometry (not necessarily a unitary) and multiplying the v_t's for $t = k2^{-n}$, $\Delta t = 2^{-n}$, may end up giving the 0 intertwiner between $(Q_0' \cap N) * \mathbb{C}$ and $\alpha_1((Q_0' \cap N) * 1) = \mathbb{C} * (Q_0' \cap N)$. But this issue was resolved in ([P2,3]) by the discovery of "symmetric paths", which allows patching intertwiners in a way that makes each incremental intertwiner have range matching the

domain of the next incremental intertwiner.

Thus, based on deformation/intertwining techniques in ([P2,3]), the only conceptual novelty in this paper is the idea of using "spectral gap rigidity" of embeddings $Q \subset N \subset M = N * N$, for non-amenable subalgebras Q, which allows to argue that any deformation by automorphisms of M is uniform on $Q' \cap N$. This idea was first used in ([P5]), but for deformations of the identity in McDuff factors $M = Q\overline{\otimes}R$ with Q non-(Γ), to prove that such factors have "unique McDuff" decomposition, up to unitary conjugacy. The proof of that result, which inspired the approach here, is presented at the end of the paper.

§2. Two lemmas

Before giving the rigorous details of the argument outlined above, let us first prove properties (1) and (2). (N.B. We will not need the last part of Lemma 1 below.)

Lemma 2.1 ([P2]). *The free group factors $N = L\mathbb{F}_n, 2 \leq n \leq \infty$ have the following property: There exist a continuous action $\alpha : \mathbb{R} \to \mathrm{Aut}(N * N)$ and a period 2 automorphism $\beta \in \mathrm{Aut}(N * N)$ such that*

(1.1). $\alpha_1(N * \mathbb{C}) = \mathbb{C} * N$.

(1.2). $N * \mathbb{C} \subset (N * N)^\beta$.

(1.3). $\beta\alpha_t\beta = \alpha_{-t}, \forall t$.

Moreover, α, β can be chosen so that to commute with all automorphisms of $N * N$ implemented by permuting the same way, simultaneously, the left and right generators of $\mathbb{F}_n * \mathbb{F}_n$.

Proof To prove the statement it is clearly sufficient to construct the continuous action α of $\mathbb{R}$ on $M = N * N$ with the period 2-automorphism β of M such that $\beta\alpha_t\beta == \alpha_{-t}, \forall t \in \mathbb{R}$ and such that there exist isomorphic copies N_1, N_2 of N in M which are mutually free, generate M and satisfy $\alpha_1(N_1) = N_2$, $N_1 \subset M^\beta$.

To this end, let $a_1, a_2, ...$ be the generators of $\mathbb{F}_n$ viewed as unitary elements in $N * \mathbb{C}$ and $b_1, b_2, ...$ the same generators but viewed as unitary elements in $\mathbb{C} * N$.

Let $h_k \in \mathbb{C} * N$ be self-adjoint elements with spectrum in $[0, 2]$ such that $b_k = \exp(\pi i h_k), \forall k$. We then put $\alpha_t(a_k) = \exp(\pi i t h_k)a_k$ and $\alpha_t(b_k) = b_k, k = 1, 2, ..., t \in \mathbb{R}$. It is trivial to see that $\exp(\pi i t h_k)a_k$ and b_k are mutually free and generate the same von Neumann algebra as a_k, b_k. Thus, α_t defines an automorphism of $N * N$, $\forall t \in \mathbb{R}$. Note also that $\alpha_1(a_k) = b_k a_k = b'_k$ is free with respect to a_k and they jointly generate the same algebra as a_k, b_k do. Thus, if we let $N_1 = N * \mathbb{C}$ and $N_2 = \alpha_1(N * \mathbb{C})$ then N_1, N_2 are free and generate M. Moreover, by the definition we clearly have $\alpha_t\alpha_s = \alpha_{t+s}, \forall t, s \in \mathbb{R}$, showing that α is a continuous action.

Define now $\beta(a_k) = a_k$ and $\beta(b_k) = b_k^*, \forall k = 1, 2, ...$ This clearly defines a period 2 automorphism of $M = N * N$ satisfying $N * \mathbb{C} \subset (N * N)^\beta = M^\beta$. Moreover

$$\beta(\alpha_t(\beta(a_k))) = \beta(\alpha_t(a_k)) = \beta(\exp(\pi i t h_k)a_k)$$

$$= \exp(\pi i t h_k)^* a_k = \exp(-\pi i t h_k) a_k = \alpha_{-t}(a_k).$$

Similarly, we get

$$\beta(\alpha_t(\beta(b_k))) = b_k = \alpha_{-t}(b_k),$$

showing that all conditions are satisfied.

□

From the next lemma we only need $\Rightarrow$ in the case $N = P = L\mathbb{F}_n$.

Lemma 2.2. *Let Q be a separable von Neumann subalgebra of a finite von Neumann algebra (N, τ) and (P, τ) a von Neumann algebra $\neq \mathbb{C}$. Then Q has no amenable direct summand if and only if, when viewing $Q = Q * \mathbb{C}$, $N = N * \mathbb{C}$ as subalgebras of $M = N * P$, we have $Q' \cap M^\omega \subset N^\omega$.*

Proof. If Q has amenable direct summand Qp, for some non-zero $p \in \mathcal{P}(Q)$, then by Connes theorem Qp is approximately finite dimensional and the statement follows from the case when Qp is finite dimensional, when we trivially have $(Qp)' \cap pM^\omega p \not\subseteq N^\omega$.

If Q doesn't have amenable direct summand but $Q' \cap M^\omega \not\subseteq N^\omega$, then let $x = (x_n)_n \in Q' \cap M^\omega$, $x \notin N^\omega$. By substracting $E_{N^\omega}(x) = (E_N(x_n))_n$ (which still commutes with Q), we may assume $x \neq 0$ but $E_N(x_n) = 0, \forall n$, thus $x_n \in L^2M \ominus L^2N$. By replacing $(x_n)_n$ with a subsequence and using the separability of Q, we may also assume $[x_n, y] \to 0$, $\forall y \in Q$. Note that if b is the conditional expectation of xx^* onto Q, then $b \in Q' \cap Q = \mathcal{Z}(Q)$ and $\lim_\omega \tau(y x_n x_n^*) = \tau(yb), \forall y \in Q$, is a normal trace on Q.

By the usual decomposition of $L^2M = L^2(N * P)$ we see that as N bimodule $L^2M \ominus L^2N$ is identical to the Hilbert space $\mathcal{H}$ obtained as the direct sum of infinitely many copies of $L^2N \overline{\otimes} L^2N$, i.e. $\mathcal{H} = L^2N \overline{\otimes} L^2N \overline{\otimes} \ell^2\mathbb{N}$. We regard each summand $L^2N \overline{\otimes} L^2N$ as Hilbert-Schmidt operators on L^2N, and view $\mathcal{H}$ as the space $HS(L^2N) \overline{\otimes} \ell^2\mathbb{N}$ of "diagonal" Hilbert-Schmidt operators in $HS(L^2N \overline{\otimes} \ell^2\mathbb{N})$, in the obvious way, with $\langle \cdot, \cdot \rangle_{HS}$ the scalar product on $\mathcal{H}$. If we view the elements x_n of the sequence $(x_n)_n$ as Hilbert-Schmidt operators X_n in $\mathcal{H}$, then $\lim_n \|uX_nu^* - X_n\|_{HS} = 0$, $\forall u \in \mathcal{U}(Q)$ and $\lim_\omega \langle yX_n, X_n \rangle_{HS} = \lim_\omega \tau(y x_n x_n^*) = \tau(yb)$. Thus, if we define $\varphi(T) \stackrel{\text{def}}{=} \lim_\omega \langle TX_n, X_n \rangle_{HS}$, $\forall T \in \mathcal{B} = \mathcal{B}(L^2N \overline{\otimes} \ell^2\mathbb{N})$ then φ is a positive functional on $\mathcal{B}$ which has Q in its centralizer and restricted to Q equals $\tau(\cdot b)$, i.e. a normal trace. Thus Q has an amenable summand, by [C]. □

§3. Proof of the main result

Proof of Theorem 1.1. Assume Q has a non-amenable direct summand. Since $N = L\mathbb{F}_n$ is a factor, by replacing if necessary Q with $m \times m$ matrices over a corner of it, we may assume Q does not have an amenable direct summand but still $Q^0 = Q' \cap N$ is diffuse. By Lemma 2.2, $\forall \varepsilon > 0$, $\exists \delta > 0, F \subset \mathcal{U}(Q)$ finite such that if $x \in (M)_1$ satisfies $\|ux - xu\|_2 \leq \delta$, $\forall u \in F$, then $\|x - E_N(x)\|_2 \leq \varepsilon$.

Let (α, β) be as in Lemma 2.1. By continuity of α there exists $t = 2^{-n}$ such that $\|\alpha_{-t}(u) - u\|_2 \leq \delta/2, \forall u \in F$. It follows that

$$\|[\alpha_{-t}(x), u]\|_2 \leq 2\|u - \alpha_{-t}(u)\|_2 \leq \delta, \forall x \in (Q^0)_1, u \in F$$

implying that $\|\alpha_{-t}(x) - E_N(\alpha_{-t}(x))\|_2 \leq \varepsilon$, $\forall x \in (Q^0)_1$, or equivalently

$$\|x - E_{\alpha_t(N)}(x)\|_2 \leq \varepsilon, \forall x \in (Q^0)_1.$$

By ([P1], or 2.1 in [P3], or [PSiSm]; a modification of arguments in [Ch] will also do), there exist projections $q \in Q^0$, $q' \in (Q^0)' \cap M$, $p \in \alpha_t(N)$, an isomorphism $\rho : qQ^0q \to p\alpha_t(N)p$, a projection $p' \in \rho(qQ^0qq')' \cap pMp$ and a partial isometry $v \in M$ such that $v^*v = qq', vv^* = pp'$, $vx = \rho(x)v, \forall x \in qQ^0q$, and v at distance $f(\varepsilon)$ from 1, where $\lim_{\varepsilon \to 0} f(\varepsilon) = 0$.

But by (Remark 2) of 6.3 in [P4]), since Q is diffuse (because non-amenable) $Q' \cap M \subset N$. Thus $q' \subset N$ so altogether $Q_0 = qQ^0qq'$ lies entirely in N. Similarly, since the image of ρ is a diffuse algebra and $\alpha_t(N)$ splits off M in a free product, by ([P4]) again it follows that $p' \in \alpha_t(N)$. We have thus just shown that there exists a diffuse subalgebra $Q_0 = qQ^0qq' \subset N$ and a partial isometry v in M such that $v^*v = 1_{Q_0}$, $vQ_0v^* \subset \alpha_t(N)$.

With n the fixed integer with $2^{-n} = t$ as before, we now construct by induction over $k \geq 0$ some partial isometries $v_k \in M$ and diffuse weakly closed von Neumann subalgebras $Q_k \subset N = N * \mathbb{C}$ such that

(a) $$\tau(v_k v_k^*) = \tau(vv^*), v_k^* v_k = 1_{Q_k}, v_k Q_k v_k^* \subset \alpha_{1/2^{n-k}}(N)$$

Letting $Q_0 = Q_0, v_0 = v$, we see that the relation holds true for $k = 0$. Assume we have constructed v_j, Q_j for $j = 0, 1, \ldots, k$. By applying the automorphism β of Lemma 2.2 to the inclusion in (a) and using the properties of β, it follows that

(b) $$\beta(v_k) Q_k \beta(v_k)^* \subset \alpha_{-1/2^{n-k}}(N)$$

By further applying $\alpha_{1/2^{n-k}}$ to this latter inclusion it follows that

(c) $$Q_{k+1} \stackrel{\text{def}}{=} \alpha_{1/2^{n-k}}(\beta(v_k))\alpha_{1/2^{n-k}}(Q_k)\alpha_{1/2^{n-k}}(\beta(v_k^*)) \subset N \otimes \mathbb{C}$$

By conjugating (c) with $\alpha_{1/2^{n-k}}(\beta(v_k^*))$ we thus get:

(d) $$\alpha_{1/2^{n-k}}(\beta(v_k^*))Q_{k+1}\alpha_{1/2^{n-k}}(\beta(v_k)) = \alpha_{1/2^{n-k}}(Q_k) \subset \alpha_{1/2^{n-k}}(N)$$

On the other hand, by applying $\alpha_{1/2^{n-k}}$ to (a) we also have:

(e) $$\alpha_{1/2^{n-k}}(v_k)\alpha_{1/2^{n-k}}(Q_k)\alpha_{1/2^{n-k}}(v_k^*) \subset \alpha_{1/2^{n-k-1}}(N)$$

Altogether, it follows that if we let $v_{k+1} = \alpha_{1/2^{n-k}}(v_k)\alpha_{1/2^{n-k}}(\beta(v_k^*))$ then by (d) and (e) we get:

$$v_{k+1}Q_{k+1}v_{k+1} \subset \alpha_{1/2^{n-k-1}}(N)$$

Moreover, since $\beta(v_k^* v_k) = v_k^* v_k$, we also have $v_{k+1}v_{k+1}^* = \alpha_{1/2^{n-k}}(v_k v_k^*)$, so that $\tau(v_{k+1}v_{k+1}^*) = \tau(v_k v_k^*)$. This ends the induction argument. Taking $k = n$, by (1.1) it follows that $v_n Q_n v_n^* \subset \alpha_1(N * \mathbb{C}) = \mathbb{C} * N$. But by (Corollary 4.3 in [P4]), this implies that $v_n = 0$. Since $\tau(v_n v_n^*) = \tau(vv^*)$, it follows that $v = 0$, a contradiction. □

§4. Some remarks

1°. In ([O]) Ozawa called *solid* a II_1 factor N with all subalgebras with diffuse relative commutant being amenable. As we mentioned in the introduction, this property implies N is prime. Primeness phenomena for II_1 factors first appeared in ([P4]), where it was shown that free group factors with uncountably many generators have this property. The natural problem of showing that separable free group factors are prime as well ([P4]), mentioned also in ([P6]), remained open for some time, until Ge solved it using Voiculescu's free entropy techniques ([Ge]). Ge then asked whether there exist diffuse subalgebras of $L\mathbb{F}_n$ with non-amenable relative commutant, a problem settled by Ozawa's result.

2°. Despite strong interest and much effort, Voiculescu's other remarkable application of free entropy to free group factors, namely the non-existence of Cartan subalgebras in $L\mathbb{F}_n$, could not be shown (so far) by the new methods of Ozawa and Peterson ([O], [Pe]), nor by Connes-Shlyakhtenko's cohomological invariants ([CS]). Our approach here may open up some new perspectives in this respect.

3°. Ozawa's property can be viewed as a "non-embeddability into $L\mathbb{F}_n$" of algebras of the form $\tilde{Q} = \overline{\mathrm{sp}}QQ'$, with Q, Q' commuting, Q non-amenable and Q' diffuse. Combining this with an argument of Connes-Jones in ([CJ]) can be used to show that any group Λ which can be decomposed as a product of two commuting subgroups $\Lambda = HH'$ with H non-amenable and H' infinite, has a properly outer cocycle action on $L\mathbb{F}_\infty$ which cannot be inner-perturbed to an actual action (i.e. the 2-cocycle involved does not vanish). Indeed, because if it could then by ([CJ]) we would have $L_\mu(\Lambda) \subset L\mathbb{F}_2$, for some scalar 2-cocycle $\mu : \Lambda \times \Lambda \to \mathbb{T}$ of the group Λ, where $L_\mu(\Lambda)$ denotes as in ([CJ]) the von Neumann algebra generated by the unitaries u_g acting on $\xi_h = (\delta_{h,k})_k \in \ell^2\Lambda$ by $u_g(\xi_h) = \mu(g,h)\xi_{gh}$, for $g, h \in \Lambda$. If H' contains torsion free elements, then it is easy to see that this entails $L_\mu(H)' \cap (L\mathbb{F}_2)^\omega$ has a diffuse part. But then (Proposition 7 in [O]) implies $L\mathbb{F}_2$ has a diffuse abelian subalgebra with non-amenable relative

commutant in $L\mathbb{F}_2$, contradicting Theorem 1. Similarly if H' contains elements of arbitrary large period. We are thus reduced to the case when there exists $n < \infty$ such that $g^n = e$, $\forall g \in H'$. This implies the 2-cocycle μ satisfies $(\mu(g,h)/\mu(h,g))^n = 1$ for all $g \in H', h \in H$. Thus, if we denote by $\{u_g \mid g \in \Lambda\}$ the canonical unitaries in $L_\mu(\Lambda)$, then for each $g \in H'$ the subgroup $H_g = \{h \in H \mid u_h u_g = u_g u_h\}$ is normal with index $\leq n$ in H, and in fact $H/H_g \hookrightarrow \mathbb{Z}/n\mathbb{Z}$. Applying this to all $g \in H'$ we get a group morphism $H \to (\mathbb{Z}/n\mathbb{Z})^{H'}$. Since the right hand group is abelian and H is non-amenable, the kernel $H_0 \subset H$ must be non-amenable. Thus $Q = L_\mu(H_0)$ is non-amenable and commutes with the diffuse algebra $L_\mu(H')$, contradicting Theorem 1.

§5. A related result

As mentioned earlier, we end the paper with the proof of a result in ([P5]), on the unique tensor product decomposition of McDuff II_1 factors, which inspired the above proof of Theorem 1.1.

Thus, let M be a factor of the form $M = P\overline{\otimes}R$ where R is a copy of the hyperfinite II_1 factor and P is a non(Γ) II_1 factor. Note that if $M_n \subset R$ are finite dimensional such that $\overline{\cup_n M_n} = R$, then the factors $P_n = P \otimes M_n$ increase to M and thus the expectations E_{P_n} converge to id_M.

Let $Q \subset M$ be another non(Γ) II_1 subfactor of M such that $M = Q \vee Q' \cap M$. By ([C]) it follows that Q has "spectral gap" with respect to $Q' \cap M$, i.e. $\forall \varepsilon > 0$, $\exists u_1, ..., u_n \in \mathcal{U}(Q)$ and $\delta > 0$ such that if $x \in (M)_1$ satisfies $\|[u_i, x]\|_2 < \delta, \forall i$, then $\|x - E_{Q'\cap M}(x)\|_2 \leq \varepsilon$.

Since $E_{P_n} \to id_M$, there exists n such that $\|E_{P_n}(u_i) - u_i\|_2 < \delta/2, \forall i$. But then for any $x' \in (P' \cap M)_1$ we have

$$\|[x', u_i]\|_2 = \|[x', u_i - E_{P_n}(u_i)]\|_2 < \delta,$$

implying that $\|E_{Q'\cap M}(x') - x'\|_2 \leq \varepsilon$. This shows that $P_n' \cap M \subset_\varepsilon Q' \cap M = R$, which implies

$$Q = (Q' \cap M)' \cap M \subset_{2\varepsilon} (P_n' \cap M)' \cap M = P_n.$$

By ([OP]) this implies that there exists $u \in \mathcal{U}(M)$ such that $uQu^* = P^t, u(Q' \cap M)u^* = R^{1/t}$, after some "re-scaling" with an appropriate $t > 0$ of the decomposition $M = P\overline{\otimes}R$, as in ([OP]). We have thus proved:

Theorem 5.1 ([P5]). *If a II_1 factor M has two decompositions of the form $M = N_1\overline{\otimes}R_1 = N_2\overline{\otimes}R_2$, with N_1, N_2 non(Γ) II_1 factors and $R_1, R_2 \simeq R$, then there exists a unitary element $u \in M$ such that $uN_1u^* = N_2^t$, $uR_1u^* = R_2^{1/t}$, for an appropriate $t > 0$.*

If one applies this to free group factors $N_1 = L\mathbb{F}_n$, $2 \leq n < \infty$, and $N_2 = L\mathbb{F}_\infty$, it follows that $L\mathbb{F}_n\overline{\otimes}R \simeq L\mathbb{F}_\infty\overline{\otimes}R$ iff $L\mathbb{F}_n\overline{\otimes}\mathcal{B}(\ell^2\mathbb{N}) \simeq L\mathbb{F}_\infty\overline{\otimes}\mathcal{B}(\ell^2\mathbb{N})$. By a result of Dykema ([Dy]) and Radulescu ([R]), this shows the following:

Corollary 5.2. *The free group factors $L(\mathbb{F}_s), 2 \leq s \leq \infty$, are all isomorphic iff they are isomorphic after "stabilizing" with R, i.e. iff $L\mathbb{F}_s\overline{\otimes}R \simeq L\mathbb{F}_\infty\overline{\otimes}R$, $\forall s \geq 2$, and iff there exists $s \geq 2$ such that $L\mathbb{F}_s\overline{\otimes}R \simeq L\mathbb{F}_\infty\overline{\otimes}R$.*

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